Term Structure of Volatility and Non-Standard Expiry

Glen Swindle

June 21, 2013

Outline

- Origins of Non-Standard Expiry
- Empirical Results
- Variations in Models
- Non-Stationarity

Non-Standard Expiration

The Issue

- Vanilla options markets in commodities always have expiration dates very near the delivery tenor
- Examples:
  - Natural Gas: Futures expiration is '-3b' and option expiration is '-4b'.
  - Crude Oil: Futures expiration is 's-m+24d-0b-3b' and options expiration is 's-m+24d-0b-6b'
- The only implied vols you see are those for options on contract T_m which span the time interval $[0, T_m^{(e)}]$ where $T_m^{(e)} \lessapprox T_m$.

Non-Standard Expiration

The Issue

- Options structures arise in which expiration varies substantially from the standard market convention.
- Modeling / estimation is required to infer implied vols over non-standard time interval from vanilla implied vols.
- Examples:
  - CSO's: In which the second leg has non-standard expiry.
  - Price-holds for potential transactions:

$$\max\left[K - V\left(F(\tau, \bar{T})\right), 0\right]$$

    where V is the value of the structure in consideration and K is the offered price.

    - * Swaptions are a subset of such derivatives. Annual swaptions expire on the value of an cal strip:

$$V = \frac{1}{12}\sum_m d(\tau, T_m)F(\tau, T_m)$$

.

Non-Standard Expiration

The Issue

- Examples (cont):
  - Options on ETFs:
    - These involve the vol of the current nearby, which requires inference from term vols for each contract to the vol pertinent to the time when the contract is active in the index.
    - The recursive dynamics of V_t is due to the contract rolls:

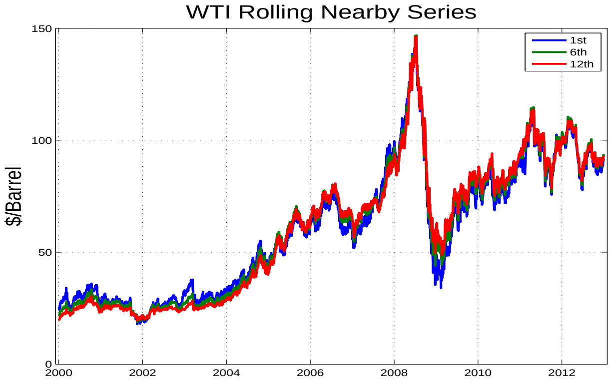
$$V_t = \left[\prod_{n=1}^{N(t)} \frac{F(T_n, T_n)}{F(T_n, T_{n+1})} \right] F(t, T_{N+1})$$

for $t \in (T_n, T_{n+1}]$ where $N(t) = \max\{n : T_n < t\}$

    - The returns volatility of the value process V is identical to the returns volatility for contract N over the time interval $t \in (T_{n-1}, T_n]$ which is non-standard.

Introduction

- Consider the following plot of WTI nearby prices
  - Clearly prices (and returns) are highly correlated.
  - Reconciliation of high returns correlations with the empirical fact that forward curves can show periods of significant backwardation requires that the volatility of forward prices decrease with tenor.



Dealing with Contract Expiration

Non-Stationarity

- Empirical analysis of price dynamics in commodities is encumbered by a technical annoyance: Contracts expire.
  - Price data is of the form $F(t_n, \bar{T}_n)$.
  - The set of active contract months depends on the trade date t.
  - The tenor in reference to trade date $\bar{T}_n - t$ changes with t.
  - Contract expiration results in discontinuities that can can corrupt analysis.

- There are two basic approaches:
  - Construct rolling nearby series.
  - Construct constant-maturity forwards.

Dealing with Contract Expiration

Nearby Constracts and Constant Maturity Forwards

- Two Methods
  - "Nearby" contracts refer to sequence of live contracts.
    - The "1st nearby" is the first live contract. The "second nearby" is the next live contract.
    - The terms "front month" and "prompt month" also refer to the "1st nearby"
  - Do not concatenate 1st nearby prices
    - This results in monthly jumps as contracts expire and does not correspond to any tradable activity.
  - Concatenate returns calculated on a common contracts
    - Integrating the series if a synthetic price is required.
- Constant Maturity (CM) Forwards
  - Interpolate the market data represented in the form: $\{\bar{T}_n - t_n\}_{n=1}^{N}$, $\{F(t_n, \bar{T}_n)\}_{n=1}^{N}$ onto a fixed grid $\bar{T}$.

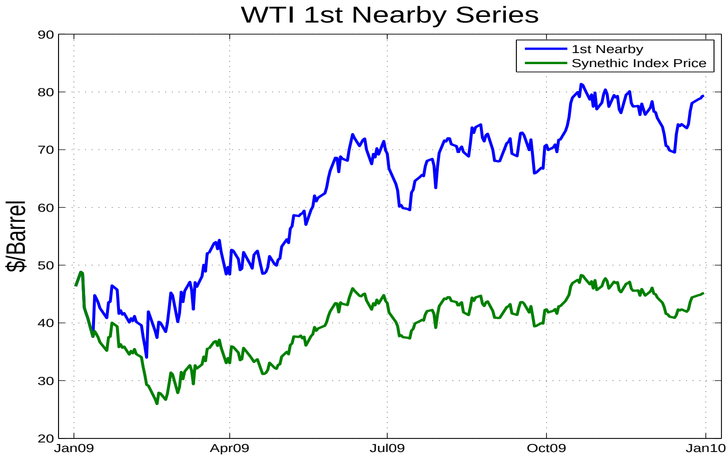
Dealing with Contract Expiration

Implications for index investors:

- The important point is that the value of the index evolves according to the changes in the current first nearby price $F(t, T_{N+1})$.
- This means that the index returns are identical to the first nearby *returns* series.
- It is the index price series that is relevant to financial investors.
- The figure below shows the price trajectory for the first nearby contract for WTI over the course of 2009 as well as the index price that an investor would have experienced by summing the first nearby returns.
- Investors that opted to go long oil via index or ETF during this period shown realized an annual return of -3%, while optically the 1st nearby price series gained over 50%.

Dealing with Contract Expiration

Implications for index investors:



Term Structure of Volatility and Correlation

Volatility Backwardation

- The next figure shows the annualized realized returns volatility by tenor using the CM approach:

$$r_{\Delta}(t, T) \equiv \log \left[\frac{F(t + \Delta, T)}{F(t, T)} \right]$$

  and data from 2005 to 2008 using $\Delta = 10$ days.

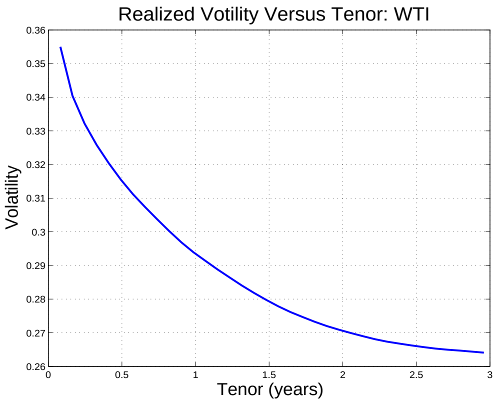
- Note the systematic decrease of returns variance with tenor, a phenomenon frequently referred to as the Samuelson effect.

- This is typical, not just of commodities forward curves, but of a variety of term structures arising in the financial markets.

Term Structure of Volatility and Correlation

Volatility Backwardation

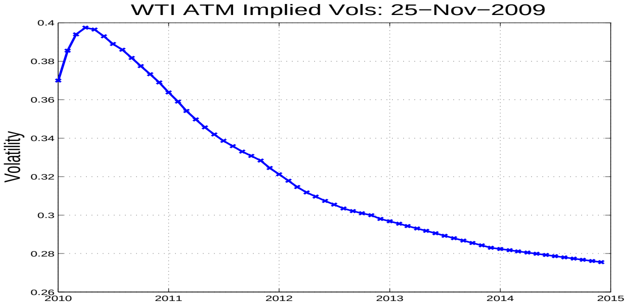
- This gives us a handle (or at least some guidance) with respect to $\sigma(T - t)$.



Term Structure of Volatility and Correlation

Volatility Backwardation

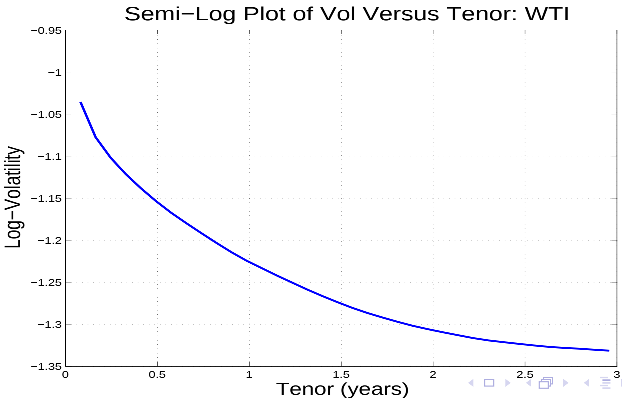
- Implied vols exhibit a similar structure.
  - The figure shows WTI ATM implied vols for European monthly options.
  - Note the backwardation in implied vols (at least at longer tenors) that is consistent with the empirical analysis.
  - The vol contango at short tenors can arise and results in nontrivial modeling questions addressed later.



Term Structure of Volatility and Correlation

Volatility Backwardation

- Exponential forms describing the decay in standard deviation of returns will turn out to be analytically convenient.
- The following plot shows a semi-log plot which suggests subexponential decay.
- A single exponential is unlikely to do much for us.

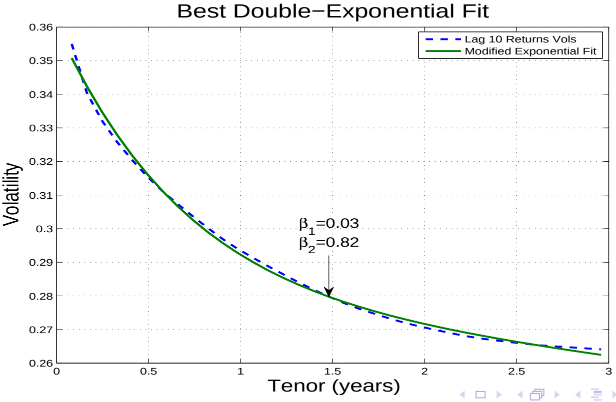


Term Structure of Volatility and Correlation

Volatility Backwardation

- Fitting with a double exponential form:

$$\sigma^2(T-t) = \sum_{j=1}^{2} \sigma_j^2 e^{-2\beta_j(T-t)} = \sigma_1^2 \left[e^{-2\beta_1(T-t)} + \lambda e^{-2\beta_2(T-t)} \right] \tag{1}$$



Non-Standard Expiration

Working Problem

- On 18Feb2009 you are asked by the sales desk to price WTI Dec11 at-the-money (ATM) European straddles with expiration on 17Dec2009 .
    - This is the simplest incarnation of non-standard expiration.
    - Most occurrences are much more complex.
- What you know
    - Dec11 is a liquid contract with standard options expiration date 16Nov2011.
    - NYMEX options markets are visible on this horizon.

Non-Standard Expiration

Market Data

- The relevant market data for this problem is shown below.

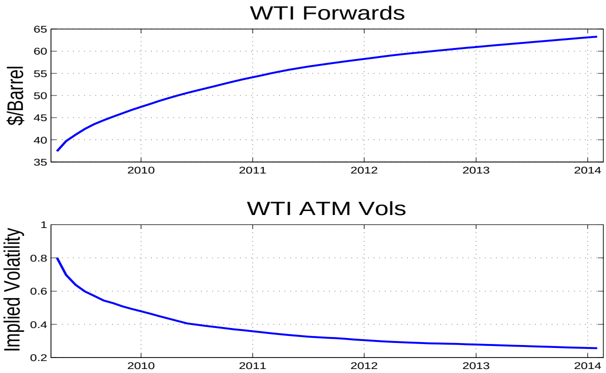


Figure: WTI Market Data (18Feb2009)

Non-Standard Expiration

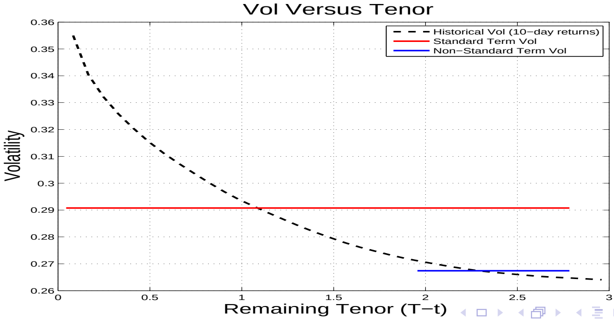
Market Data

- The relevant times for this problem are:
  - Dec11 contract expiry: $T = 2.784$.
  - Dec11 standard option expiry: $T^{(e)} = 2.740$
  - Non-standard option expiry: $T^* = 0.827$.
- The market data directly pertinent to this problem consist of:
  - The forward price: $F(0, T) = 57.970$.
  - The implied vol for the standard option: $\bar{\sigma}_{T^{(e)}} = 0.308$.

Non-Standard Expiration

Backwardation of Volatility

- The näive approach of applying Black valuation using $\bar{\sigma}$ over the interval $[0, T^*]$ would seem to have the potential of over valuing the option due the backwardation of volatility.
  - The dashed curve shows shows historical returns volatility by tenor from Jan2005 to Dec2008.
  - The horizontal lines show the term volatilities over $[0, T^{(e)}]$ and $[0, T^*]$.
  - The separation in the horizontal lines of over two vol points illustrates a potential problem with this approach.



Single-Contract Time-Varying Vol

Modification of Black

- One possible approach is the following generalization of Black:

$$\frac{dF_t}{F_t} = \sigma(T - t)dB_t$$

where F_t denotes the Z11 forward price in our example.

- The forward value for the option (i.e. setting $r = 0$) is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(T - t)^2 F^2 \frac{\partial^2 V}{\partial F^2} = 0$$

.

- The first order of business is to select a reasonable $\sigma(T - t)$ for which the solution to the previous Black PDE yields the known market value implied by $\bar{\sigma}$.

Single-Contract Time-Varying Vol

Solution by Time Change

- To value any payoff expiring at $\tau \in [0, T]$ the standard method is to construct a time change $s(t)$ so that:

$$\frac{\partial V}{\partial s} + \frac{1}{2}\bar{\sigma}_\tau^2 F^2 \frac{\partial^2 V}{\partial F^2} = 0$$

  and:

  - $s(0) = 0$
  - $s(\tau) = \tau$.

- Standard Black valuation with constant vol σ_τ holds.

Single-Contract Time-Varying Vol

Solution by Time Change

- Time change:

$$s(t) = \tau - \frac{\int_t^\tau \sigma^2(T-u)du}{\bar{\sigma}_\tau^2}$$

where

$$\bar{\sigma}_\tau^2 = \frac{1}{\tau}\int_0^\tau \sigma^2(T-u)du$$

does the job.

- In particular noting that:

$$\frac{ds}{dt} = \frac{\sigma^2(T-t)}{\bar{\sigma}_\tau^2}$$

changes the original time-varying PDE into the constant vol PDE.

- The punchline is that time varying vol is equivalent to using the average vol over the valuation period $[0, \tau]$.

Single-Contract Time-Varying Vol

Solution by Time Change

- To summarize: Valuing an option expiring at τ in the time-varying vol setting, is equivalent to standard Black:

$$\frac{\partial V}{\partial s} + \frac{1}{2}\bar{\sigma}_\tau^2 F^2 \frac{\partial^2 V}{\partial F^2} = 0$$

  using a variance equal to the average variance of the time-varying vol over the valuation interval $[0, \tau]$:

$$\bar{\sigma}_\tau^2 = \frac{1}{\tau}\int_0^\tau \sigma^2(T - u)du$$

Single-Contract Time-Varying Vol

Parameterization and Calibration

- Using our parametric estimate for $\sigma(t)$, we now select σ_1 so that $\sigma(T-t)$ is consistent with the market implied vol $\sigma_{T^{(e)}}$.
- Given the time-change results, this amounts to finding σ_1 so that:

$$\begin{aligned}
\bar{\sigma}^2_{T^{(e)}} &= \frac{1}{T^{(e)}} \int_0^{Te} \sigma^2(T-u)du \\
&= \frac{1}{T^{(e)}} \int_0^{Te} \sum_{j=1}^{2} \sigma_j^2 e^{-2\beta_j(T-u)} du \\
&= \frac{1}{T^{(e)}} \sigma_1^2 \left[\Psi\left(T, T^{(e)} | \beta_1\right) + \lambda \Psi\left(T, T^{(e)} | \beta_2\right) \right]
\end{aligned}$$

where we have simplified appearances by defining:

$$\Psi\left(T, S | \beta_j\right) \equiv \frac{e^{-2\beta_j(T-S)} - e^{-2\beta_j T}}{2\beta_j}$$

Single-Contract Time-Varying Vol

Parameterization and Calibration

- Solving for σ_1 yields:

$$\sigma_1^2 = \frac{\bar{\sigma}^2_{T^{(e)}} T^{(e)}}{\left[\Psi\left(T, T^{(e)} | \beta_1\right) + \lambda \Psi\left(T, T^{(e)} | \beta_2\right)\right]}$$

- Using this value of σ_1 results in consistency with the market implied vol $\bar{\sigma}_{T^{(e)}}$.

- Calibration is complete.

Single-Contract Time-Varying Vol

Parameterization and Calibration

- We now use the calibrated parametric form to calculate the implied vol applicable to the interval $[0, T^*]$: $\bar{\sigma}_{T^*}$ by integrating:

$$\bar{\sigma}^2_{T^*} = \frac{1}{T^*}\int_0^{T^*} \sigma^2(T-u)du$$

.

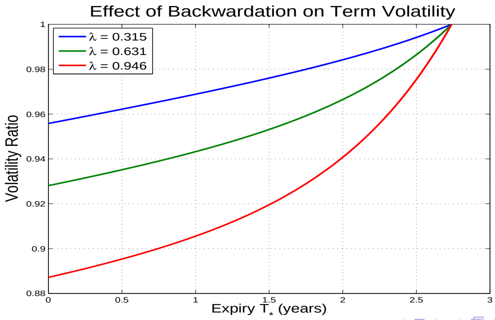
- The result can be expressed as a ratio of the non-standard variance to the vanilla variance:

$$\frac{\bar{\sigma}^2_{T^*}}{\bar{\sigma}^2_{T^{(e)}}} = \frac{T^{(e)}}{T^*}\left[\frac{\Psi\left(T, T^*|\beta_1\right) + \lambda\Psi\left(T, T^*|\beta_2\right)}{\Psi\left(T, T^{(e)}|\beta_1\right) + \lambda\Psi\left(T, T^{(e)}|\beta_2\right)}\right] \tag{2}$$

Single-Contract Time-Varying Vol

Qualitative Effects

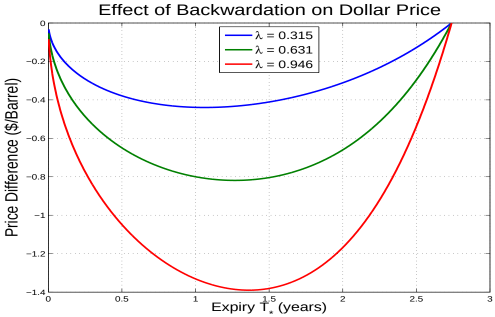
- The next figure shows the effect of backwardation rate on the ratio of implied volatilities.
- Note that the higher the value of λ the more backwardated the local volatility and the greater the effect of early expiry.



Single-Contract Time-Varying Vol

Qualitative Effects

- Ultimately the dollar price of the options is what matters in life.



Single-Contract Time-Varying Vol

Working Problem Solution

- Using the parameters of the original working problem the relevant volatility ratio is 0.922.
- This results in an implied vol $\bar{\sigma}_{T^*} = 0.284$ and a mid-market value of \$11.908/MMBtu obtained by using the basic Black valuation formulas with this implied vol and all other market data as specified.
- Contrast this to the näive Black value of \$12.916/MMBtu.

Single-Contract Time-Varying Vol

Hedging

- The implied volatility $\bar{\sigma}_{T^*}$ used to obtain our mid-market value was calculated using:
  - The double exponential parameterization.
  - The implied volatility of the vanilla option $\bar{\sigma}_{T^{(e)}}$.
- Analysis of hedge performance involves not merely neutralizing exposure to the value of $\bar{\sigma}_{T^{(e)}}$ but also analysis of the effect of variations in the parametric form.
- We will proceed by assuming that the estimated parameters $\bar{\pi} \equiv [\beta_1, \beta_2, \lambda]$ are constant.

Single-Contract Time-Varying Vol

Hedging: Vega

- Assume that we are long the non-standard option with value function V.
- At any given time t we will hold a quantity $q(t)$ of an option with standard expiry $T^{(e)}$ (and value function U) so as to neutralize:

$$\frac{\partial(V+qU)}{\partial\bar{\sigma}_{T^{(e)}}} = \frac{\partial V}{\partial\bar{\sigma}_{T^*}} \frac{\partial\bar{\sigma}_{T^*}}{\partial\bar{\sigma}_{T^{(e)}}}\Big|_{\bar{\pi}} + q\frac{\partial U}{\partial\bar{\sigma}_{T^{(e)}}}|_{\bar{\pi}} \tag{3}$$

- The first term arises by application of the chain rule yielding the product of the conventional vega for the non-standard option and the sensitivity of the term vol $\bar{\sigma}_{T^*}$ to the market implied vol $\bar{\sigma}_{T^{(e)}}$ assuming that $\bar{\pi}$ is constant.

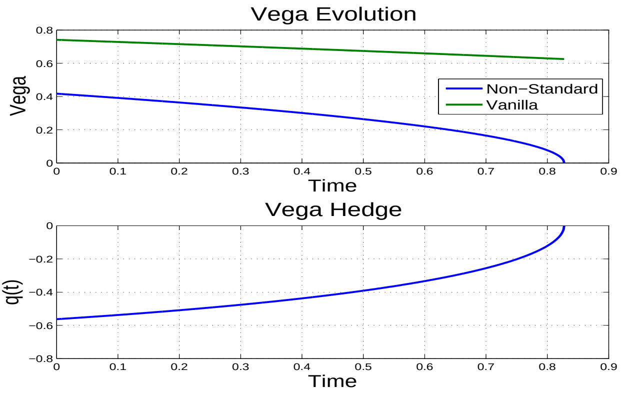
Single-Contract Time-Varying Vol

Hedging: Vega

- Constructing a vega hedge is now merely arithmetic.
- The following figure shows the evolution of vega assuming constant underlying price $F(t, T)$ for our two options as well as the evolution of $q(t)$.
- Clearly this vega hedge will require repeated rebalancings of the short vanilla option hedge, which can result in a rather large cumulative transaction costs in practice.
- In some markets, notably power, the significant bid/offer spreads in the options market would seriously impede effecting anything but the coarsest approximation to this hedging strategy

Single-Contract Time-Varying Vol

Hedging: Vega



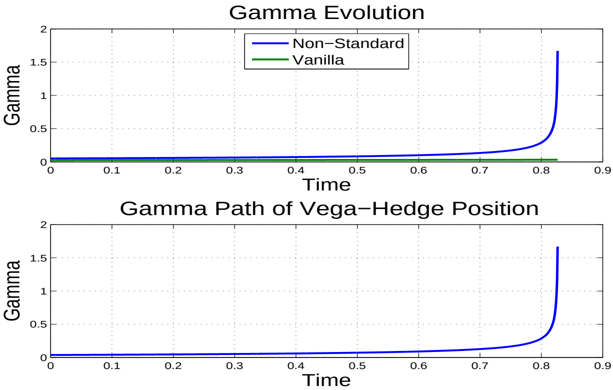
Single-Contract Time-Varying Vol

Hedging: Vega

- A second issue also emerges in this working problem: The vanilla options market consists essentially exclusively of options that expire near the delivery month.
- As a consequence, under this modeling paradigm it is not possible to simultaneously eliminate vega and gamma risks.
- The following figure shows the evolution of gamma for the two options as well as the resulting gamma path for the vega-hedged portfolio under the same assumption of unchanged forward price.
- Note the severe divergence of the total gamma $t \uparrow T^*$.
- This motivates the question: was a vega hedge of the risk variable $\bar{\sigma}_{T^{(e)}}$ the right thing to do?

Single-Contract Time-Varying Vol

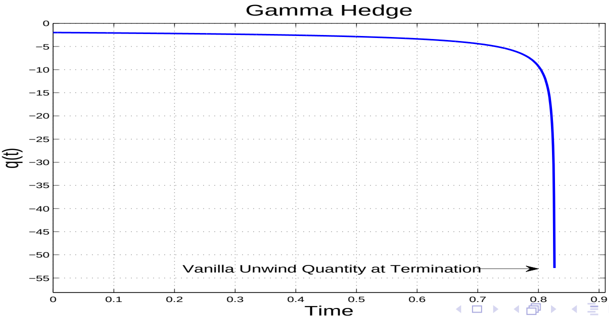
Hedging



Single-Contract Time-Varying Vol

Hedging: Gamma

- Suppose instead we opt to maintain zero gamma.
  - The next figure shows the vanilla hedge quantity to maintain gamma neutrality on the constant forward path.
  - This shows the "dual" problem where large vega exposures can arise.
  - Near expiry very large quantities of the vanilla option need to be held (and unwound at expiry).



Broader Modeling Paradigms

Big-T Calibration

- Viewing all contracts simultaneously, what we did in the previous approach is calibrate each contract (independently of the other contracts) "around" our parametric form for vol term structure.

$$\frac{dF(t,T)}{F(t,T)} = \sigma(T)\Phi(T-t)dB_t$$

where

$$\Phi(T-t) \equiv \left[e^{-2\beta_1(T-t)} + \lambda e^{-2\beta_2(T-t)}\right]^{\frac{1}{2}}$$

  - The function $\sigma(T)$ was calibrated contract by contract.
  - The implied volatility of a contract $\bar{\sigma}_{T_j}$ effects only $\sigma(T_j)$ and no other $\sigma(T_k)$ $k \neq j$.

- All that we have done here is couple the dynamics of all contracts.

Broader Modeling Paradigms

little-t Calibration

- An alternative is to allow the local volatility to be a function of current time t which we will refer to as "little-t" calibration:

$$\frac{dF(t,T)}{F(t,T)} = s(t)\Phi(T-t)dB_t$$

- It is convenient (and standard) to assume a piecewise constant form for $v(t)$:

$$s(t) = s_j \mathbf{1}_{\{T_{j-1} < t \leq T_j\}}$$

Broader Modeling Paradigms

Heuristics

- little-t calibration assumes that local perturbations effect the entire set of forward returns.
- If circumstances transpire to effect the implied volatility for near-dated contracts longer tenors will follow suit.
- There are numerous "fundamental" reasons to justify the presence of the little-t effect.
  - Hurricane season in the Gulf of Mexico peaks in late summer and has the recurring potential to disrupt both supply and demand of oil and natural gas in North America.
  - The appearance of a tropical disturbance off of the coast of Africa, which increases the probability of a hurricane a week later in the Gulf of Mexico, should (and does) increase implied vols across all tenors.
  - Given storability of each of these commodities, one would expect the local vol profile to be increased over the upcoming week(s), not uniformly enhanced over the life term options.

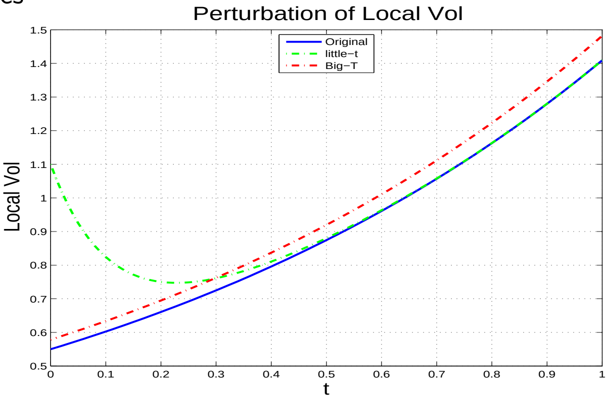
Broader Modeling Paradigms

Heuristics

- The following figure depicts the two calibration approaches in such a situation.
- The plots show the model-estimated instantaneous vol as a function of spot time t through the evolution of a single contract.
- Note that "Big-T" calibration "interprets" information with a shift of this estimated vol through the life of the contract, in contrast to "little-t" which perturbs the vol near $t = 0$.

Broader Modeling Paradigms

Heuristics



Broader Modeling Paradigms

Heuristics

- This motivates the potential value of a hybrid approach:

$$\frac{dF(t,T)}{F(t,T)} = s(t)\Phi(T-t)\sigma(T)dB_t$$

which we will discuss briefly later.

Time-Varying Local Volatility

little-t Calibration

- We now return to calibration of the little-t approach and construct hedging strategies using the results.
- The fact that the local volatilities are functions of t means that:
  - The implied vol $\bar{\sigma}_{T_j}$ is a function of $\left[s_1, \ldots, s_j\right]$.
  - The value of an implied vol $\bar{\sigma}_{T_j}$ effects the local vols s_k for $k > j$ via a bootstrap calibration.

Time-Varying Local Volatility

little-t Calibration

- Using the same double-exponential parameterization of vol backwardation, calibration requires:

$$\bar{\sigma}_J^2 T_J^{(e)} = \sum_{j=1}^{J} s_j^2 \int_{T_{j-1}^{(e)}}^{T_j^{(e)}} \left[e^{-2\beta_1 (T_J - t)} + \lambda e^{-2\beta_2 (T_J - t)} \right] dt$$

- Intuition:
  - During each time interval $\left[T_{j-1}^{(e)}, T_j^{(e)}\right]$ the local vol s_j is effecting all contracts in play.
  - A higher s_j means that all contracts experience a higher vol.
  - The exponential terms means, however, that local vol backwardates. The realized vol for contract k is lower than that for contract j if $T_k > T_j$ due to the exponential terms.

Time-Varying Local Volatility

little-t Calibration

- The result is the bootstrap solution for calibration:

$$s_J^2 = \frac{\bar{\sigma}_J^2 T_J^{(e)} - \sum_{j=1}^{J-1} s_j^2 \mathcal{I}\left[T_{j-1}^{(e)}, T_j^{(e)}, T_J\right]}{\mathcal{I}\left[T_{J-1}^{(e)}, T_J^{(e)}, T_J\right]}$$

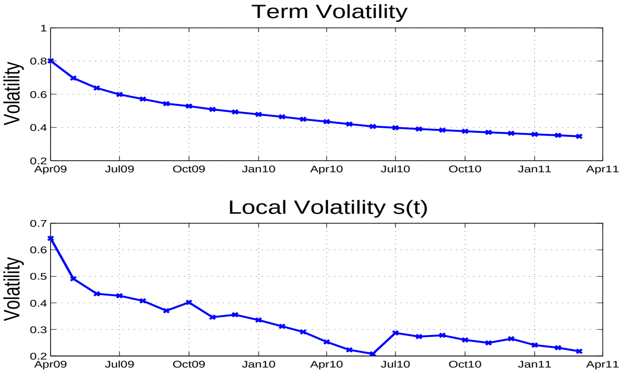
  where we have defined:

$$\mathcal{I}\left[t_1, t_2, T\right] \equiv \int_{t_1}^{t_2} \left[e^{-2\beta_1(T-t)} + \lambda e^{-2\beta_2(T-t)}\right] dt$$

Time-Varying Local Volatility

Calibration

- The result for our working problem is shown below.



Time-Varying Local Volatility

Calibration

- Note that the local vol $s(t)$ is rapidly decreasing in t.
  - The trade date for this problem happened to be in a period of extreme levels of both volatility in absolute, as well as volatility backwardation in crude markets in the immediate aftermath of the credit crisis.

- The roughness of $s(t)$ is due to kinks in the marked implied vol, often barely visible. This is an artifact of simple interpolation methods by traders or exchanges between relative liquid contracts.

Time-Varying Local Volatility

Working Problem Solution

- The implied volatility for our Z11 contract over $[0, T^*]$ is obtained by summing the contributions of each of the intervening intervals.
  - The interval $[0, T^*]$ is decomposed into:

    $$0 < T_1^{(e)} < T_2^{(e)} \ldots < T_{J(T^*)}^{(e)} < T^*$$

    where J^* denotes the last contract with expiration before T^*.
  - The expression for $\bar{\sigma}_{T^*}$ below simply sums the variances over each interval:

    $$\bar{\sigma}_{T^*}^2 T^* = s_1^2 \mathcal{I}\left[0, T_1^{(e)}, T\right] + \ldots + s_{J^*}^2 \mathcal{I}\left[T_{J^*-1}^{(e)}, T_{J^*}^{(e)}, T\right] + s_{J^*+1}^2 \mathcal{I}\left[T_{J^*}^{(e)}, T^*, T\right]$$

  - Note that T above is referring to the tenor of the Z11 contract.

Time-Varying Local Volatility

Working Problem Solution

- This yields a term vol $\bar{\sigma}_{T^*} = 0.404$ and a value of \$16.892.
- Contrast this to the näive Black value of \$12.916/MMBtu and well as the Big-T result \$11.908/MMBtu.
- The results are in complete contrast to the previous Big-T approach. The high values of $s(t)$ near $t = 0$ imply a high volatility for *all* contracts.

Time-Varying Local Volatility

Hedging

- In constructing a vega hedge, we now have exposure to s_j for every contract j: $1 \leq j \leq J^*$.
- An exact vega hedge amounts to constructing a portfolio so that exposure to s_j for all $j \leq J^*$ is zero.
- In what follows we will assume that we are using vanilla ATM straddles for contracts $\bar{T} \equiv \left[T_1, \ldots, T_{J(T^*)}\right]$ with value functions $\bar{U}$.

Time-Varying Local Volatility

Hedging

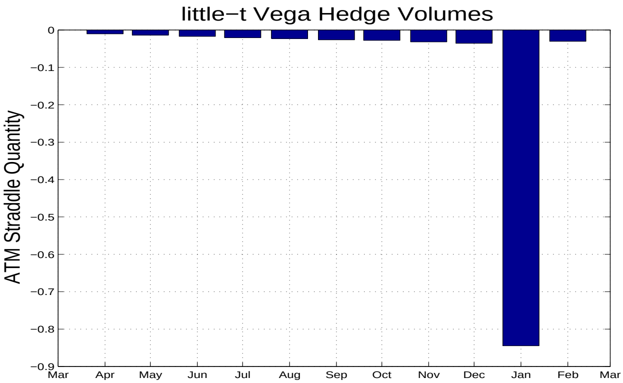
- Denoting the hedge quantity by $\bar{q}$, which must solve the linear system:

$$A\bar{q} = -\bar{v} \qquad \text{where} \qquad A_{i,j} = \frac{\partial U_j}{\partial s_i} \qquad \text{and} \qquad v_i = \frac{\partial V}{\partial s_i}$$

- Note that A is lower triangular as the exposure of the j^{th} option does not have exposure to an s_i for $i > j$.
- The next figure shows the options positions required.
  - The last active contract J^* corresponds to the Jan10 (F10) contract.
  - In fact, the option expiry of this contract is 17Dec2009 which coincides with the expiry of our nonstandard option problem.

Time-Varying Local Volatility

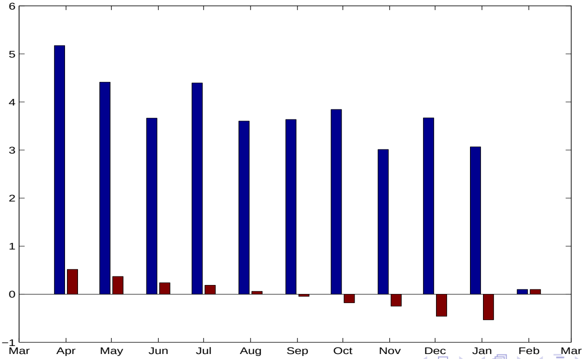
Hedging



Time-Varying Local Volatility

Hedging

- The dominance of the last option, which is most "matched" to our non-standard option from an expiry perspective, motivates a much simpler hedge: using only the last relevant contract.
- The resulting vega slippage is shown below and is clearly quite acceptable.



Conclusion

Big-T

- For a given parametric form for backwardation of vol $\Phi(T-t)$, calibration of implied vols is done contract by contract.
- Advantage: Calibration is unambiguous and Greeks pertain only to each relevant contract.
- Disadvantage: No information about other contract implied vols is used in valuation of a derivative on a specific contract.
  - This is at variance with empirical results (PCA).
  - This restricts possible hedging strategies and can result in severe gamma or vega divergences.

Conclusion

little-t

- For a given parametric form for backwardation of vol $\Phi(T - t)$, calibration of involves bootstrap to generate local vols $\bar{s}$.
- Advantage: Calibration utilizes information from all contracts.
  - This is reasonable given empirical results.
  - This facilitates using options across contracts for hedge construction and mitigates divergences.
- Disadvantage: For a given form of Φ calibration can fail.

Conclusion

little-t

- The intuition behind the failure of calibration at say contract k is that high levels of term vol $\bar{\sigma}_j$ for $j < k$ can result in a value of integrated variance for contract k that exceeds that implied by the term variance $\bar{\sigma}_k^2 T_k^{(e)}$.

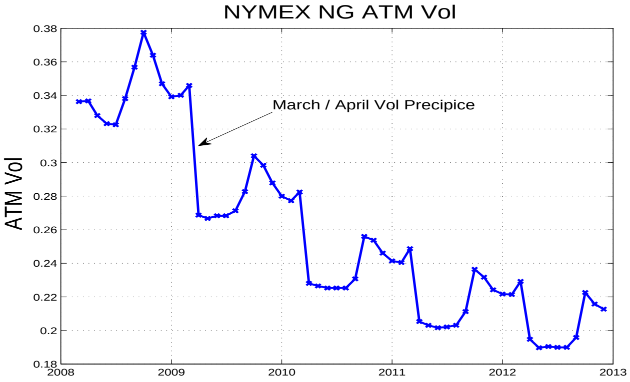
$$\sum_{j=1}^{k-1} \int_{T_{j-1}}^{T_j} \sigma_j^2 e^{-2\beta(T_j - s)} ds > T_k \bar{\sigma}_k^2$$

- The local vol in time interval k would have to contribute a negative variance correction; something that is obviously not possible.

Conclusion

little-t

- The following figure shows a sample of the term structure of ATM vol for NG.



Conclusion

little-t

- To clarify this point, suppose for the moment we choose the simpler form for local vol backwardation:

$$\Phi^2(T-t) = e^{-2\beta(T-t)}$$

- Letting $\mu(T) = \bar{\sigma}_T^2 T$ and $\zeta(t) = s^2(t)$, calibration amounts to solving:

$$\mu(T) = \int_0^T \zeta(t) e^{-2\beta(T-t)} dt$$

Conclusion

little-t

- Differentiating this with respect to T yields:

$$\frac{\partial \mu(T)}{\partial T} = \zeta(T) - 2\beta \int_0^T \zeta(t) e^{-2\beta(T-t)} dt$$

- If the implied vol $\bar{\sigma}(T)$ decreases at a sufficient rate, $\frac{\partial \mu(T)}{\partial T} < 0$.
- If we reduce β to a sufficiently low level (this amounts to choosing a parametric form which backwardates slowly enough) the second term on the right-hand-side will be small enough in magnitude so that we must have $\zeta(T) < 0$; clearly an impossibility.

Conclusion

little-t

- This is origin of failure to calibrate, which can arise in situations where implied vols backwardate sufficiently rapidly relative to ones choice of $\Phi(\cdot)$.
- The implication is that when this happens the market implied vols are inconsistent with the form of Φ.
- Either a faster backwardation or use of a hybrid approach utilizing non-constant values of σ. would be required.

Conclusion

Complications

- Energy options markets do not resolve the form of local volatility due to inherent limitations in liquid tradables.
- Which of these two approaches (or any hybrid) is a modeler's choice, guided by econometrics.
- The world is non-stationary.

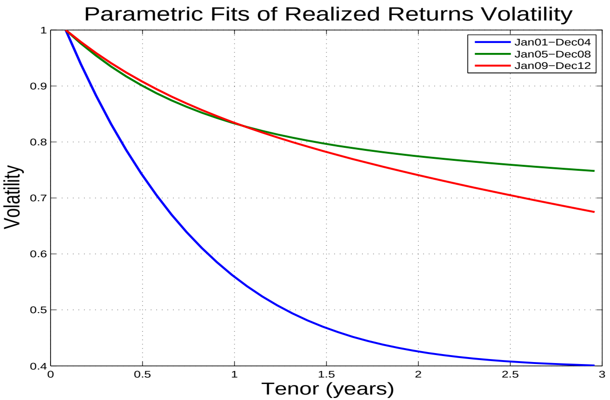
Conclusion

Evolving Market Dynamics

- Stability of the double-exponential fit an issue, and one that is hard to hedge.
- The results for the minimum L^2 error using 10-day returns fit shown below using data starting at three different non-overlapping 4 year intervals with starting dates: Jan2001, Jan2005 and Jan2009.

Conclusion

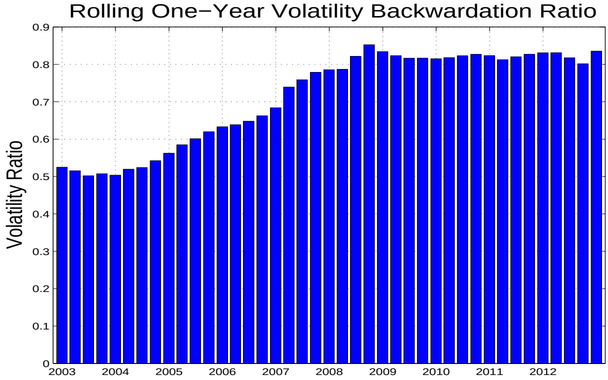
Evolving Market Dynamics



Conclusion

Evolving Market Dynamics

- The following shows rolling 4 year estimate of the 1 year volatility normalized by the 0 year vol, indexed by end date of estimation interval.
- As this and the previous figures show, the variation in estimated backwardation is significant.



Summary

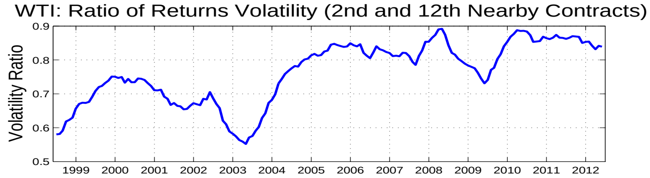
Evolving Market Dynamics

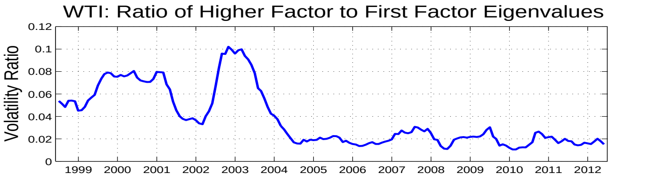
- The dynamics of the WTI forward curve has become increasingly one-dimensional in nature.
- In the next figure:
  - The top plot displays the ratio of the rolling one-year realized volatility for the 12th nearby contract returns to that of the 2nd nearby, with the results plotted versus the mid-point of the calendar averaging window.
  - The lower plot shows the ratio of higher order total volatility to the that of the first factor: $\frac{\left(\sum_{j=2}^{24} \lambda_j\right)^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}}$ from PCA analysis of the first 24 nearby contracts, applied over the same rolling one-year windows.

Summary

Evolving Market Dynamics

- Understanding (let alone predicting) such behavior remains an open problem.





On Deck

- Application of Multi-Factor Models
- Lessons From Tolling Deals